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Moving Geometry from the Back of the Book:

A Report on Geometry and Measurement in the 1986 Assessment

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Our thanks to the MEAP Mathematics Curriculum Advisory Committee:

Marie T. Appleby	South Hadley Middle School, South Hadley
Paul L. Bartolomucci	Minuteman Vocational-Technical School, Lexington
Thomas Carroll	Chelmsford Public Schools, Chelmsford
Parker Damon	McCarthy-Towne School, Acton
Ellen Holway	Green Meadow School, Maynard
Paul Lyons	Cambridge Public Schools, Cambridge
Therese M. McKillop	Braintree Public Schools, Braintree
Diana Nunnaley	Walsh Middle School, Framingham
Thomas A. Risoldi	Salem High School, Salem
Winston J. Rose	Masconomet Regional School, Topsfield
Barbara Shea	Blanchard Memorial School, Boxborough
Christine Thompson	Center School, Leicester
Patricia Tremblay	Umana High School, Boston
John J. Waite Jr.	Winchester High School, Winchester
Claire J. Zalewski	Framingham State College, Framingham

Introduction

In April 1986 the Massachusetts Department of Education assessed the performance of all third, seventh, and eleventh grade children in mathematics, science, and reading. The majority of items used in the assessment were drawn from the National Assessment of Educational Progress; however, these items were supplemented by others in order to reflect more local emphasis.

In addition to overall scores in mathematics, more detailed scores were reported according to type of content and according to the kind of thought process demanded by different items. As was the case in the overall score, the percentage of correct responses to the items in each of these reporting categories was transformed to a scaled score with a statewide average of 1300 and standard deviation of 100 points.

Scaled scores are an excellent device for comparing performance of different schools or performance of the same school in different topic areas. Another type of information of use to schools requires going back to the items themselves and examining the actual responses of students.

The purpose of this report is to give schools a more accurate picture of students' achievement in geometry and measurement. Geometry was chosen as our focus for a number of reasons. The first reason has to do with the value of the subject itself. Primarily, the Mathematics Curriculum Committee believed that, despite its fundamental importance to mathematical thinking, geometry is a topic that is largely ignored in elementary and middle schools. Furthermore, even when the subject is recognized as a formal part of the curriculum in the secondary school, the excessive formalism that often surrounds it obscures its importance in understanding and dealing with the world around us.

This observation is not limited to Massachusetts. When eighth grade achievement in geometry and measurement was tested in 20 countries throughout the world, the United States scored well below the international average. Its scores were, in fact, among the lowest quarter of all countries. At the twelfth grade, among college preparatory students, results were even more disappointing. Although, in the latter case, poor results may be explained in part by the curriculum content, at the earlier grade it was found that little time is spent in teaching geometry, relative either to other mathematical topics or to its instructional emphasis in other countries.

This report is divided into five sections. Chapter One gives an overview of the subject as a whole. It presents the case for the increasing importance of geometry instruction in the curriculum. Chapter Two reviews the results of the MEAP on geometry and measurement items. Chapter Three addresses itself to classroom teachers in the elementary and middle grades. We present some activities that can be used to increase students' understanding and preparation for future work. In Chapter Four, two mathematics educators turn their attention to the secondary curriculum. Finally, in Chapter Five, we discuss the implications of our findings for teachers and policy makers.

CHAPTER ONE: OVERVIEW

I. Introduction

We live in a visual world. We are surrounded by visual information. In fact, eighty-five percent of all information that we receive comes from the environment and enters through the visual system. The right part of our brain, the side that deals with spatial information, develops first. Only later do we begin to verbalize and to categorize information into symbolic form.

Despite these predispositions, our students find great difficulty in the one area of schooling that uses this visual world as the basis for thinking. This report concerns students' performance on visual material in mathematics. It focuses on the 1986 Massachusetts Educational Assessment items which address this area and discusses the factors that may affect this performance.

Unlike arithmetic or algebra, geometry evokes no clear-cut delineation of course material in the curriculum. The most extreme example of this comes from the Second International Mathematics Study in which only two or three of the 40 geometry items that were proposed were agreed upon as appropriate by all countries. Even in our own country there is not a clear connection between the spatial activities that fall under the rubric of geometry in the early grades and the training in deductive proof that dominates the secondary school curriculum.

Rather than assuming a consensus where none exists, in this report we will consider geometry as multi-faceted. From each viewpoint we will examine student performance, attempt to understand the causes of success and failure, and discuss implications for policy, curriculum and teaching. Finally, we will bring together some activities that teachers have found helpful in encouraging the development of geometric understanding among their students.

II. Looking at Geometry

1. Geometry as visualization: a way of understanding and ordering the world about us.

Although, for the most part, our information about the world is visual, our understanding is cognitive. From earliest days, we learn to structure our sensory impressions, to focus and to interpret. Language helps as it defines and delineates different aspects of the visual field. So does training and experience. The interior designer may see hundreds of different colors where we see only "red." The radiologist is able to distinguish significant irregularities were we see only variations of gray. In this sense, we must learn in order to see.

But there is another aspect to perception. We must also learn to interpret the language of visual expression. Art historians, anthropologists and psychologists have demonstrated that representations of three-dimensional space are not intuitively understood. For example, techniques regarding perspective and other devices for translating the three-dimensional world onto a plane surface are culturally dependent and must be learned. These techniques function as symbols, the product of highly artificial conventions. Like all symbols, they are powerful, allowing us to move beyond the concrete, to manipulate and to experiment. But to interpret them and use them, they must be learned through experience.

2. Geometry as a mathematical system which has its roots in the world about us and becomes increasingly formal and logical through its various levels of abstraction.

Geometry was the first branch of mathematics to be organized logically and, for many, Euclid's *Elements* remains the quintessential definition of the subject. However, even within this structure, there is a growing recognition that the stress on formal proof has distorted our purpose for studying the system itself.

One developmental framework proposed by two Dutch educators, the van Hieles, has received increasing empirical support. These mathematicians have suggested that there are distinct levels of geometric thought, beginning with the perception and then recognition of properties of simple figures, proceeding through the ability to interrelate those properties within and among figures, and culminating in deductive logic. Unlike most developmental schemas, the van Hieles believed that advancement depended more upon content and instructional methods than upon age.

3. Geometry as an avenue for problem solving.

For many, mathematical problem solving means computation embedded in a verbal context. And yet, the elements of problem solving — comparing, contrasting, abstracting, using inductive and deductive reasoning — are as appropriate to representational as to verbal contexts. In fact, a number of our most respected reasoning tests deliberately rely upon visual material for their content. In the classroom, on the other hand, children are often denied the opportunity to succeed in problem solving because of their deficiencies in reading or computation.

There are two contexts for geometric problem solving. One is practical, mainly involving measurement. Unlike many computational problems, measurement problems can have a constant relevance for functioning in the world around us. The other context is theoretical, involving spatial relationships of plane and three-dimensional figures. Not only does this kind of problem solving promote inductive and deductive reasoning, it presents rare opportunities for students to "think like mathematicians."

4. Geometry as a vehicle for representing mathematical or other concepts whose origin is not visual or physical.

The visual aspects of geometry allow it to be used to represent concepts and operations in other mathematical systems. For example, the number line is a useful vehicle for representing order and continuity as well as the operations of addition and subtraction. Multiplication and division can be more easily represented and understood through surface area. Through coordinate graphing, complex data can be reduced and presented in a structured manner. Proportional thinking can be represented through drawing in various scales using ideas of transformations and similarity. In order to solve many of the applied problems in calculus, the ability to construct a pictorial representation of some geometric configuration (e.g., conics, rigid figures) is essential. In the evaluation of an integral, the actual integrals encountered are often very simple to evaluate. It is finding the appropriate integrand and limits of integration that poses the greatest difficulty for the student. This, in turn, depends upon the accuracy and detail of the student's representation of the problem situation. The real problem is often one of geometry, not calculus.

5. Geometry as measurement: the tool by which we can gather information about the world around us.

Practically, we must be able to measure in order to describe, compare or contrast objects in the physical world. But this instrumental need implies more than some quantitative skill. Measurement of plane figures involves geometric concepts such as congruence. Decomposition and recomposition are often called for in the measurement of area. An understanding of the properties of a triangle is necessary for linear projections. Although geometry encompasses far more than measurement, geometric concepts are embedded in the practice of measuring.

CHAPTER TWO:
RESULTS OF THE MEAP ASSESSMENT

1. Geometry as visualization: a way of understanding and ordering the visual world about us.

We live in a three-dimensional, constantly shifting world that does not easily lend itself to the limitations of paper and pencil. However, in order to make sense of this world, students must acquire an easy familiarity with how objects occupy and move in space. Not only must they be able to recognize shapes, but they should be able to manipulate them visually: to decompose them, to enlarge them, rotate them, flip them over and move them around. The capability of visualizing the effect of action on either real or imagined objects gives the student an important advantage in later work in geometry, measurement, calculus and mathematical problem solving of all kinds.

Third grade students performed well in identifying similar shaped figures and examples of common shapes. Almost all were able to pick out examples of squares, rectangles, triangles and cubes. Most were able to identify obvious examples of non-symmetrical figures or to identify the letters that "can be folded so that both sides match." However, in contrast to their success at recognizing of symmetrical figures, approximately half of the third grade students did not appear to know how actions such as paper folding as in Figure 1 could produce symmetry.

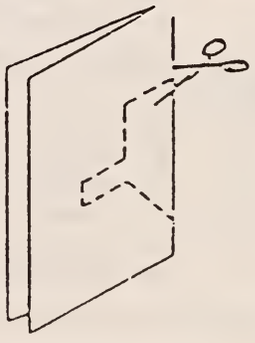
Figure 1.

Which letter can be folded so that both sides match?

% Gr. 3		
84	A.	A
4	B.	G
4	C.	F
3	D.	J
3	E.	P

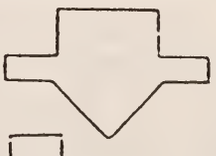
Figure 2.

What will the figure above look like when it is cut out and unfolded? Fill in the circle beside the one you choose.



% Gr. 3

15 O

29 O 

49 O

50 O

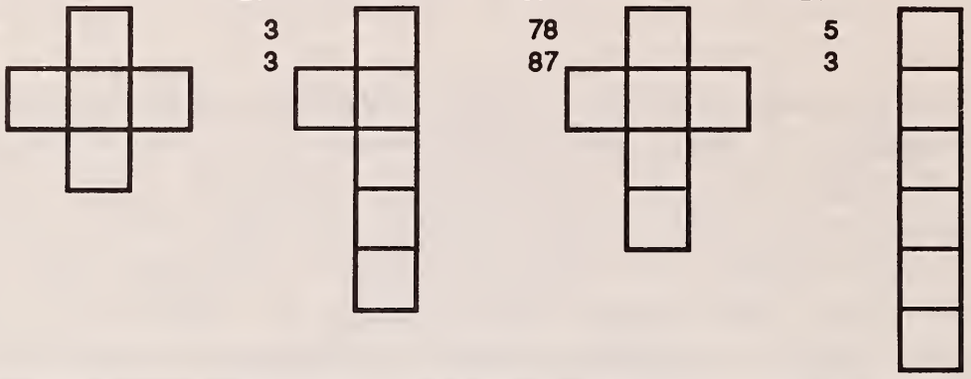
These results suggest that children may not engage in activities that help them visualize the results of operations. Supporting evidence comes from the teachers' questionnaire in which only 33% of third grade teachers indicated that their students had engaged in paper folding activities. As a result, for many children, their knowledge of symmetry and other properties of figures may be rooted in the recognition of the *results* of movement rather than in *experience itself.*

This pattern repeats itself at the older grades. A very high proportion of students recognized prototypical examples of plane figures and a common network of a cube.

Figure 3.

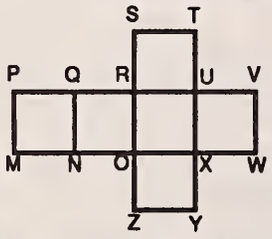
One of the pieces of wrapping paper below will fold along the lines to completely cover ALL 6 faces of this box. Which piece of wrapping paper is it?

	A.	B.	C.	D.
% Grade 7	14	3	78	5
% Grade 11	7	3	87	3



Contrast these results to those achieved when the task went beyond simple recognition. In these cases, the success rate fell dramatically.

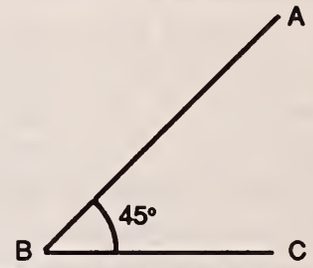
Figure 4.



The diagram shows a cardboard cube which has been cut along some edges and folded out flat. If it is folded to again make the cube, which two corners will touch corner P?

% Gr. 7	
12	A. corners Q and S
17	B. corners T and Y
13	C. corners W and Y
50	D. corners T and V
7	E. corners U and Y
2	No Response

Figure 5.



Suppose $\overline{AB}$ is rotated in space around $\overline{BC}$ so that $\angle ABC$ is always 45°. What solid figure is outlined by the path of the rotating line?

% Gr. 11	
38	A. cone
9	B. cylinder
10	C. spiral
28	D. circle
11	E. sphere
3	No Response

At grade 7, when students were presented with a problem that precluded the mere identification of a familiar network, only half were able to visualize the results of folding paper to produce a cube. (See Figure 4.) At grade 11, only 38% could recognize the result of the rotation in Figure 5. Given a set of networks and asked "Which of the patterns can be folded along the dotted lines to form an open box?", only 40% of the eleventh graders went beyond the obvious and most familiar network presented in Figure 3 to correctly identify another possibility.

Response to our questionnaire suggests that these visualization activities do not form an important part of geometry instruction. For example, when seventh grade math teachers were asked if their students had been taught the knowledge necessary to answer Figure 4, only 20% responded positively, while 31% stated that it was not in the curriculum. These summary figures masked the fact that, even when such activities are considered to be part of the curriculum, they are regarded as suitable primarily for more advanced students. Twenty-seven percent of teachers of above-average students replied that the activity was included, while only 15% of teachers of remedial students replied in the affirmative.

2. Geometry as a mathematical system which has its roots in the world about us and becomes increasingly formal and logical through its various levels of abstraction.

According to the van Hiele model, geometric ideas and understanding are built slowly through experience and instruction. The roots of abstract reasoning and deduction that define geometry as a mathematical system lie in an understanding of the properties of spatial figures and the relationship among those properties. Geometric understanding develops in stages, with each stage building upon the experiences and understanding of the preceding stages until students are able to deal with the rigor of formal deductive logic.

The first level in this developmental scheme is the recognition of common geometric figures. This ability to identify and classify figures on the basis of their general form is followed by an understanding of their specific properties, those attributes that define a circle or square, that account for circularity and squareness. A more advanced stage is reached when children are able to relate the properties of, for example, a parallelogram and a rectangle. The understanding of this relationship among properties forms the intuitive basis for the more formal definitions of such relationships.

Recognition of Shapes

Much of early geometric teaching is devoted to the recognition of simple shapes and children performed correspondingly well. Over 90% of third graders correctly identified rectangles, squares and triangles. Sixty-one percent of third graders were able to name a cylinder.

Although 81% of seventh graders and 92% of eleventh graders correctly identified a cylinder, fewer were able to relate the figure to an object in the world about them (Figure 6).

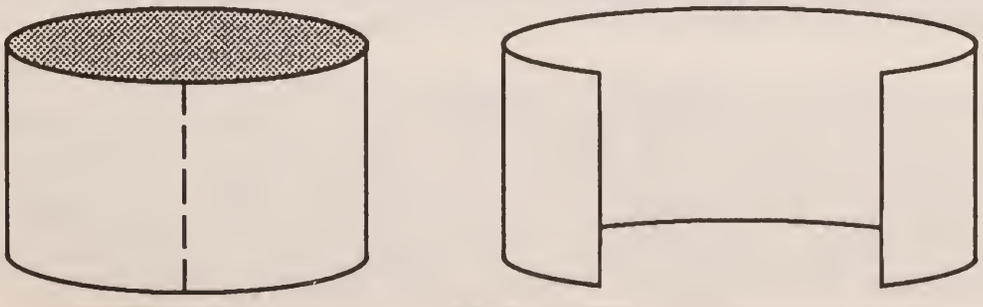
Figure 6.

Which geometric figure has a shape **most** like a drinking straw?

% Gr. 7	% Gr. 11		
6	3	A.	cone
77	87	B.	cylinder
2	1	C.	pyramid
4	2	D.	rectangular solid
10	6	E.	sphere

As in other cases, students also experienced difficulty when asked to recognize the results of an operation. For example, 30% of seventh graders were unable to visualize the figure that would result when one cut and flattened the sides of a cylinder. This inability to understand the basic components of a figure has consequences in later geometric understanding.

Figure 7.

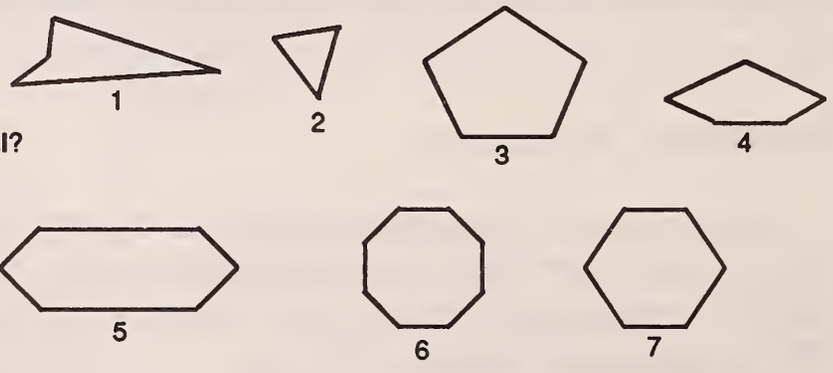


Properties of Figures

According to the developmental model of geometric ideas, a recognition of the properties of figures is the second level, following global recognition. Researchers have found that the majority of students do not progress beyond these two levels, and our assessment supports this contention.

Although, as noted earlier, students could identify prototype exemplars of common plane figures, performance declined when they were asked questions that relied upon a knowledge of properties. For example, only approximately half of seventh and eleventh graders were able to recognize irregular four-sided figures as quadrilaterals (Figure 8). In another item asking students to identify the figure which was *not* a parallelogram, 30% of seventh graders responded that a square was not a parallelogram, indicating that they did not understand what defines a parallelogram, i.e., its properties.

Figure 8.



% Gr. 11	Which figure(s) shows a **Quadrilateral?**
8	A. figures 1 and 4
15	B. figures 3, 6, and 7
46	C. figure 1 only
6	D. all of the figures
24	E. none of the figures
1	No Response

When explicitly asked about the properties of a parallelogram (Figure9), only 31% of seventh graders (53% of eleventh graders) recognized that opposite angles must be equal. In contrast to these results, 47% of seventh grade teachers stated that their students had been taught the information necessary to answer this item.

Figure 9.

% Gr. 7	% Gr. 11	Which of the following is true for all parallelograms?
31	53	A. Opposite angles are equal.
29	18	B. All sides are equal.
18	16	C. The diagonals are equal.
20	13	D. Adjacent sides are perpendicular.

This low level of performance did not pertain only to parallelograms. Only 18% of seventh graders knew that a figure with four sides of equal length must also have a right angle in order to be called a square. Twenty-seven percent of seventh graders believed that four equal sides formed a sufficient condition. Another 26% believed that parallel opposite sides was the only other requirement.

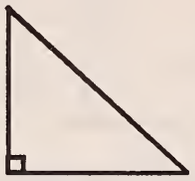
Many of the items that measured students' understanding of the properties of figures were tested concurrently by the National Assessment for Educational Progress and, therefore, cannot be reproduced for examination here. However, results clearly indicate that many students do not go beyond the recognition of figures to an understanding of the properties of those figures. Clearly, this understanding is not a major concern

of most mathematics curricula in the middle grades. Fewer than half of the seventh grade teachers responded that their students had been taught to identify the necessary properties of a parallelogram. This had been included in the curricula of only 37% of remedial classes.

However, there are additional reasons for poor performance that lie in the concepts themselves. In discussing the difficulties students experience in defining properties of spatial figures, Alan Bishop, an English mathematics educator, points out the potential confusion between form (*how* the spatial ideas are represented) and content (*what* is being represented). The fact that the concept (triangle, square, rectangle) is usually represented in only one form encourages the notion that form and content are the same.

For example,

Here is a right triangle:

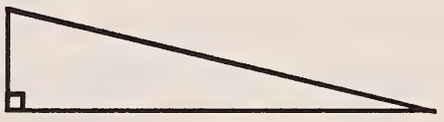


NOT

Here is a right triangle:

OR

Here is a right triangle:



This initial focus on a single exemplar hinders students from developing a general idea of what it is that defines a right triangle. In other words, the form itself captures students' attention to such a degree that they cannot go beyond the specific example to search for its general properties. As a result, students can recognize the prototypes of geometric figures, but fail to discover the defining properties of the figure.

Bishop suggests that the confusion that is common to middle and secondary school students begins in the early grades. Here, instead of concentrating on the definitions of a circle, a square and a rectangle, more attention should be paid to the geometric adjectives (circular, flat, rectangular) which can characterize the properties of objects in the child's environment.

[Bishop, Alan. "What are some obstacles to learning geometry?" In *Studies in Mathematics Education*, edited by Robert Morris. New York: UNESCO, 1986.]

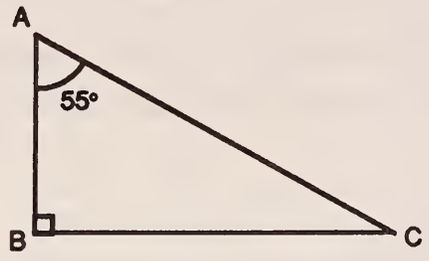
Interrelationship Between Properties of Figures

In view of their difficulty in understanding the properties of figures, it is not surprising that students performed poorly when asked to relate properties to each other.

For example, 36% of seventh graders answered correctly that the sum of the interior angles of a triangle was equal to 180°. Approximately the same percent responded that, if a triangle has two equal sides, it follows that two angles must be equal. In general, this success rate doubled by Grade 11, reflecting the fact that 71% of the students at this level had or were taking a course in geometry (20% in trigonometry).

However, despite the relatively large proportion of students who had been exposed to instruction, many eleventh grade students experienced great difficulty in applying their knowledge to solve problems. For example, in the following problem, fewer than half were able to calculate the missing angle. Are the results due to the fact that students do not know the measure of a right angle? Did they not understand the angle symbol? Do they find it difficult to reason in a geometric context?

Figure 10.

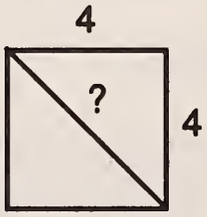


ABC is a right triangle. What is the measure of ∠ACB?

% Gr. 11	
49	A. 35°
10	B. 45°
10	C. 55°
15	D. 90°
14	E. not enough information given
1	No Response

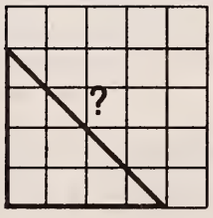
In general, eleventh grade students experienced difficulty with all but the most factual of questions. While almost two-thirds could identify supplementary angles and recognize the relationship between supplementary and complementary angles, far fewer could apply knowledge in an unfamiliar context. Given the length of the sides of a square, only 45% were able to estimate the length of its diagonal.

Example:



More than 4?
Equal to 4?
Less than 4?
Not enough information given?

When the same question was embedded in problem context, using a grid, another 10% failed to give the correct answer. Thirty-five percent counted the vertical coordinates through which the diagonal passes; another 16% counted along the legs of the right angle and added the results.



Reasoning About Relationships

The formal deductive aspects of geometry were not tested in the assessment, however, some informal deduction was included.

Compare results on the following two items which appeared in the grade 11 test. The question in Figure 11 asked students to recall the conditions for congruence and to identify the statement that violates those conditions (i.e., that two congruent figures could have different areas). The question in Figure 12 also calls upon students' knowledge of the conditions for congruence, but instead of identifying a condition that

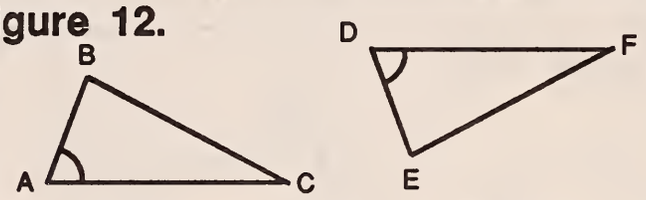
is contradictory to the concept, the student must evaluate each of the options to determine whether it is a *necessary* condition. This more abstract procedure is not only a more difficult task in reasoning but the heavy reliance of the item on symbolic representation may have discouraged students.

Figure 11.

Two triangles are congruent. Which one of the following statements about the two triangles is false?

% Gr. 11		
12	A.	The three sides of one triangle have the same lengths as the three sides of the other triangle.
11	B.	The three angles of one triangle have the same measures as the three angles of the other triangle.
61	C.	The triangles could have different areas.
16	D.	Each triangle could be placed so that it would fit exactly on top of the other triangle.
1		No Response

Figure 12.



In the triangles above, we know that $\angle A = \angle D$. Which of the following additional conditions would NOT guarantee that the two triangles are congruent?

% Gr. 11		
26	A.	$\angle B = \angle E$ and $\angle C = \angle F$
23	B.	$\angle B = \angle E$ and $AB = DE$
24	C.	$AC = DF$ and $\angle C = \angle F$
25	D.	$AB = DE$ and $AC = DF$
2		No Response

Results were slightly better when the requirement for a counter-example was presented in a more familiar context, with visual rather than symbolic exemplars. In a secure item, students were given the following statement:

If a 4-sided figure has equal diagonals, it is a rectangle.

They were then asked to identify a counter-example from an array of figures. Forty-one percent succeeded in recognizing the correct counter-example. Although these results are higher than in the previous examples, unlike the items in figures 11 and 12 this question required little geometric knowledge. The incorrect options that were chosen indicate that at least half the students *did not even recognize the premise* (i.e., that the figure must be 4-sided and have equal diagonals), and chose either a 5-sided figure or a figure with unequal diagonals as a counter-example to disprove the statement.

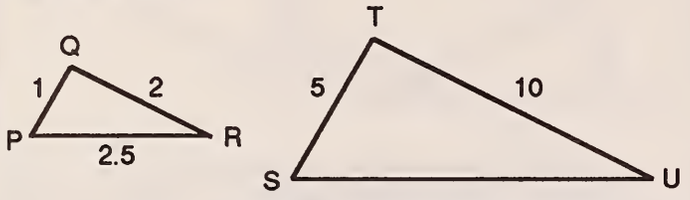
These results signify more than a lack of geometric knowledge. They suggest a lack of confidence in the ability to reason, an unwillingness to examine the information given, perhaps a fear of geometric shapes and symbols.

3. Geometry as a vehicle for representing mathematical or other concepts whose origin is not visual or physical.

The ability to think in proportional terms is fundamental to applying mathematics to the world about us. It has also proved to be a conceptually difficult skill to achieve, even among college mathematics students. However, not all proportional thinking need be based on computation. Geometric figures provide graphical representations of proportionality, while geometric concepts such as similarity and enlargement provide intuitive understanding.

Most students appeared to understand the basic principles of proportionality when applied to a simple figure, as in Figure 13, which is a simple enlargement with a factor of 5. Approximately two-thirds of eleventh grade students recognized this and answered correctly.

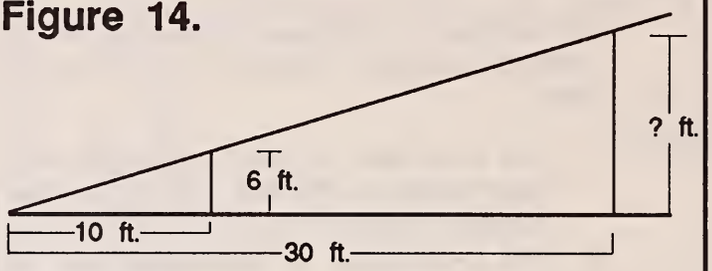
Figure 13.



Triangles PQR and STU are similar. How long is $\overline{SU}$?

%		
Gr. 11		
9	A.	5
7	B.	10
65	C.	12.5
14	D.	15
4	E.	25
1	No Response	

Figure 14.

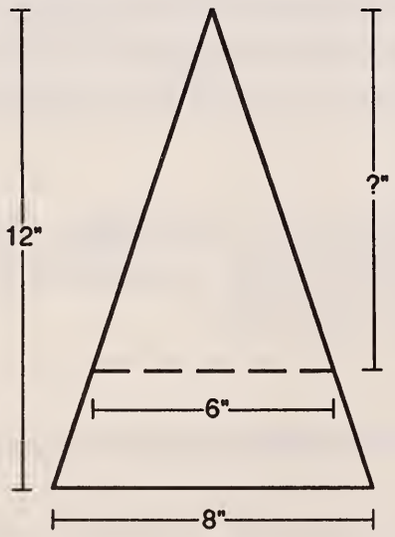


The picture above shows how Jose used a short tree to find the height of the tall tree. What answer should Jose get?

%		
Gr. 11		
8	A.	12 feet
66	B.	18 feet
17	C.	36 feet
9	D.	46 feet
1	No Response	

Figure 14 represents the same concept in a problem context. Again, two-thirds recognized that the enlarged altitude was 3 times the original. However, in Figure 15, the students were asked for the results of a reduction, and this presented them with a great deal of difficulty. Only 25% answered correctly, while almost 40% failed to recognize the proportional nature of the question and subtracted one term from another. In addition to this problem involving a reduction instead of an enlargement, it involved a more complex ratio (2:3 vs 1:3).

Figure 15.



Mrs. Watson cut out the triangular piece of cardboard above to use as a decorative Christmas tree. But she only wanted the base of the tree to be 6 inches across. If she makes the horizontal cut shown above by the dotted line, how tall will the smaller tree be?

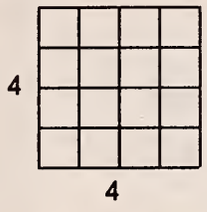
%		
Gr. 11		
16	A.	6 inches
19	B.	8 inches
25	C.	9 inches
39	D.	10 inches
1	No Response	

4. Geometry as measurement

In many senses, measurement is the application of geometric concepts to real world problems. The ability to decompose and recompose figures permits surface measurement. An understanding of the relationships of geometric properties allows for linear measurement, as seen in Figure 14 above. In addition, measurement can be used as a visual model for numerical concepts, such as associativity and commutativity.

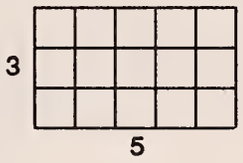
The relating of area to the counting of unit squares proved difficult for most third grade children. When presented with Figure 16, only 35% chose a rectangle with the same area as the exemplar, while 39% believed that none of the choices presented had an equivalent area.

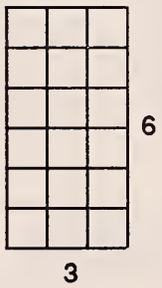
Figure 16.

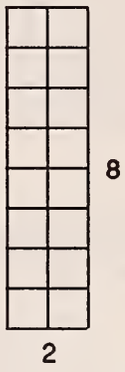


Which one of the figures to the right has the same area as the figure above?

% Grade 3	18	5	35	39
	O	O	O	O



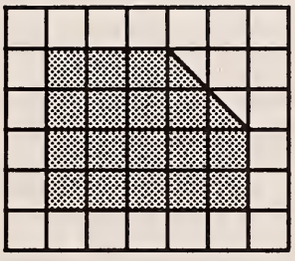




none of these

These results reflect instruction and experience, as well as cognitive difficulty. When presented with Figure 17 and asked if the information necessary for answering the question had been taught, only 19% of third grade teachers replied in the affirmative. Although these teachers may have been reacting to the part squares that were used in the figure rather than to the concept of area itself, student results on those items that stress informal exploration of area suggest otherwise.

Figure 17.

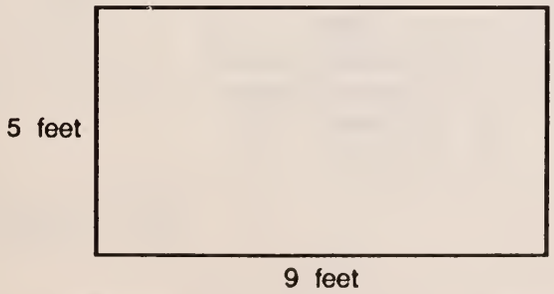


What is the area of the shaded figure?

% Gr. 3		
3	O	16
12	O	17
26	O	18
49	O	19
8	O	20

Third grade students also had difficulty with linear measurement, even when the term *perimeter* was not used (Figure 18). When asked to calculate the measure all the way around the rectangular garden, 38% of the students merely added the two numbers given.

Figure 18.

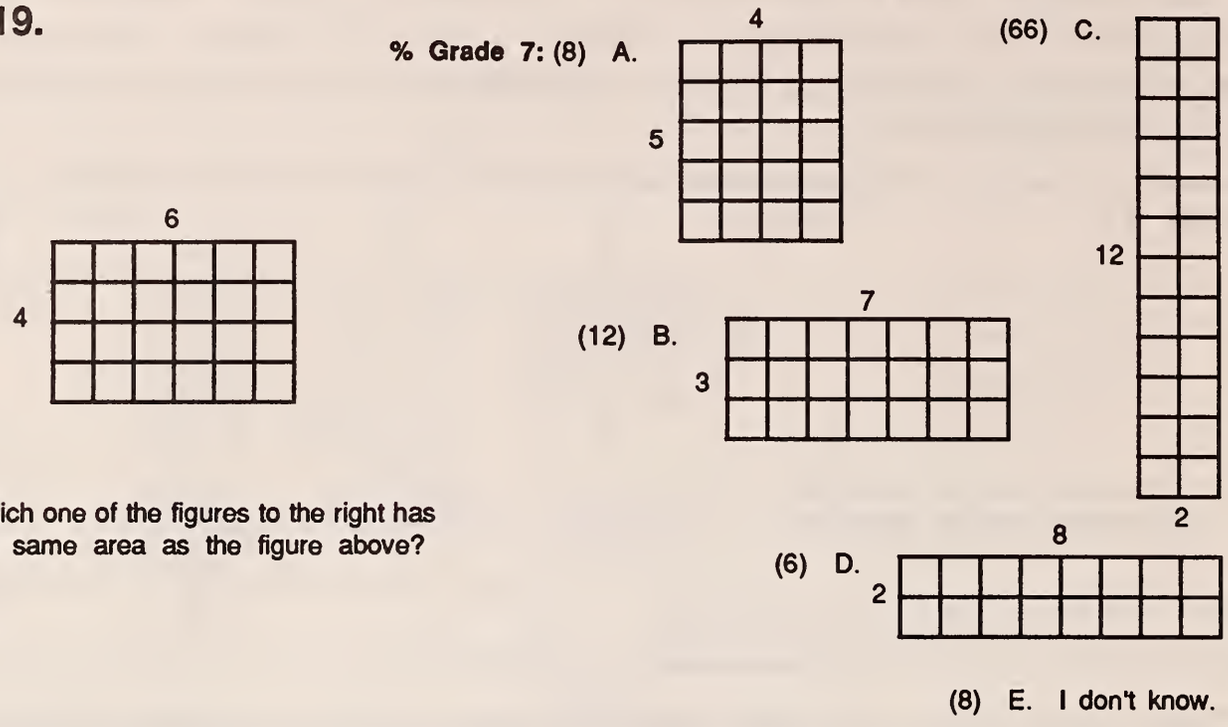


Mr. Simmons put a wire fence *all the way around* his **rectangular** garden. The garden is nine feet long and five feet wide. How many feet of fencing did he use?

% Gr. 3		
38	O	14 feet
9	O	23 feet
10	O	28 feet
28	O	45 feet

This lack of understanding of the concept of area continues into the older grades. When, at grade 7, students were presented with a problem that was almost identical to that given at grade 3, more succeeded in identifying the correct figure, but results were surprisingly poor.

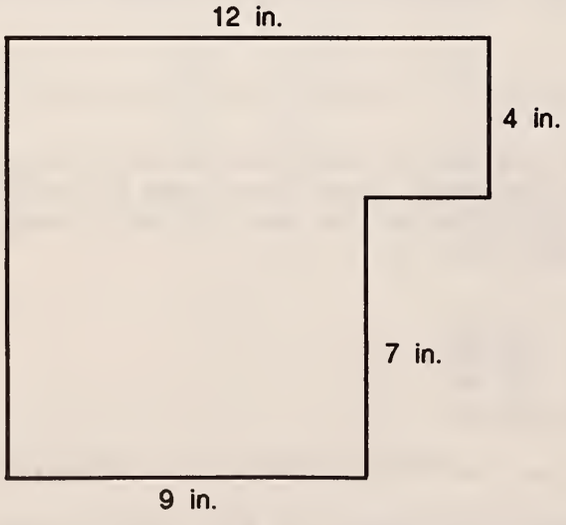
Figure 19.



A similar pattern occurred when students were presented with an irregular figure on a centimeter grid and were asked for the approximate area of the figure. Given the fact that no computation and no decomposition were required, the results suggest that students do not understand the concept itself and have not had the experiences in surface measurement activities to develop the concept.

Again, the teacher questionnaire supports the contention that area concepts are not universally taught at the seventh grade level or before. When presented with the following figure, only 43% of the teachers indicated that their seventh grade students had received the necessary information to answer the question correctly. Again, there were large discrepancies according to the type of class. Only 35% of remedial classes had received such instruction, in contrast to 52% of advanced classes.

Figure 20.

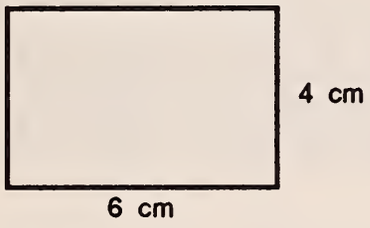


% Gr. 7	% Gr. 11	What is the area of this figure?
30	22	A. 46 inches
23	47	B. 111 square inches
31	18	C. 2 feet, 8 inches
14	12	D. 23 square inches
1	1	No Response

In order to answer the question presented in Figure 20, students could not rely on a simple process of *multiplying length times width*. Rather, they were required to partition the figure. Faced with this problem, most students did not know how to proceed. A substantial proportion added all the measures given and converted them into feet and inches. Others appeared to add some sides (12 + 4 + 7) or added and doubled.

Partially, this is a result of a lack of understanding of the basic concept of area as a covering of a surface with standard units. (See results in Figure 19.) When presented with only the linear measurements, as in Figure 21, slightly fewer than half of the seventh graders and three-quarters of eleventh graders recognized the correct procedure.

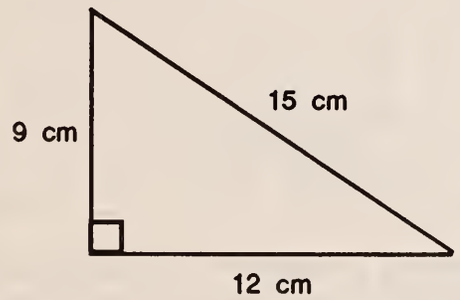
Figure 21.



% Gr. 7	% Gr. 11	What is the area of this rectangle?
2	0	A. 4 square cm
2	0	B. 6 square cm
17	5	C. 10 square cm
30	17	D. 20 square cm
46	76	E. 24 square cm
4	1	F. I don't know.

Students' difficulty with area calculations also stems from their poor understanding of the properties of plane figures and the relationship among different figures. For example, only 15% of seventh graders and 36% of eleventh graders were able to give the correct area of a right triangle, given the measurements of the legs. Few recognized the figure as half of a square. Instead, 43% of seventh graders and 26% of eleventh graders added the dimensions. Another 15-16% multiplied one dimension by another.

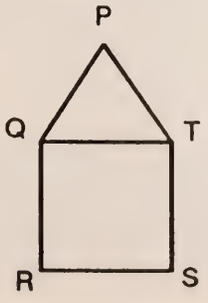
Figure 22.



% Gr. 7	% Gr. 11	What is the area of this right triangle?
11	12	A. 1620 cm^2
6	6	B. 810 cm^2
15	16	C. 108 cm^2
15	36	D. 54 cm^2
43	26	E. 36 cm^2
7	3	F. 18 cm^2
3	2	No Response

Although three-quarters of the eleventh graders could find the area of the rectangle in Figure 21, the success rate fell to 59% on the area problem in Figure 23. The latter question involved the area of a square, but was presented in a nonroutine manner.

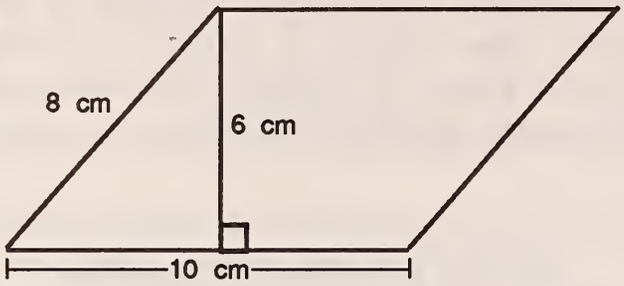
Figure 23.



% Gr. 11	The figure QRST is a square and PQT an equilateral triangle. If PQ = 6 cm then the area of the square is
4	A. 72 cm^2
6	B. 48 cm^2
6	C. 40 cm^2
59	D. 36 cm^2
24	E. 24 cm^2

This inability to relate unfamiliar figures to familiar ones was also evident in their response to a question asking for the area of a parallelogram. (See Figure 24.) In this case, they did not use the diagram to visualize its relationship to a triangle nor did they use their knowledge of the properties of a parallelogram.

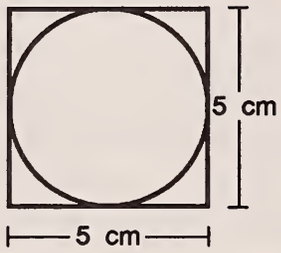
Figure 24.



% Gr. 7	What is the area of the above parallelogram?
12	A. 30 cm^2
21	B. 36 cm^2
23	C. 48 cm^2
21	D. 60 cm^2
23	E. 80 cm^2
1	No Response

Finally, although 80% of seventh graders and 89% of eleventh graders were able to identify the diameter of a circle, far fewer were able to use this knowledge in relating the properties of a circle to that of a square. In the two items below, only 30-40% of students solved the problem correctly.

Figure 25.



From the diagram above, which is the best estimate of the circumference of the circle?

% Gr. 7	% Gr. 11	
30	28	A. less than 20 centimeters
39	38	B. about 20 centimeters
24	23	C. more than 20 centimeters
N.A.	9	D. I don't know.
7	8	No Response

Figure 26.

In the diagram above, the circle is inscribed in the square. The area of the square is 25 square meters. What is the length in meters of the diameter of the circle?

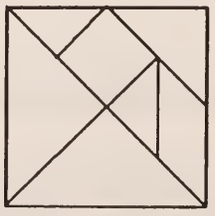
% Gr. 7	% Gr. 11	
27	41	A. 5
14	11	B. 10
25	14	C. 20
33	19	D. 25
	15	E. I don't know.
1	2	No Response

Most of the measurement items were basically reasoning problems that depended upon flexibility of thought and a problem solving approach to a far greater extent than specific knowledge of a formula or procedure. However, students' lack of knowledge of the basic properties of figures appeared to inhibit their confidence and ability to reason from the information given.

CHAPTER THREE: GEOMETRY ACTIVITIES FOR THE ELEMENTARY AND MIDDLE GRADES

It is not unusual for teachers of geometry to engage their students in entertaining activities that involve aspects of the geometry concepts they are teaching. However, sometimes the instructional potential of such activities is not fully realized. One of the greatest problems of traditional mathematics instruction is its result in the inability of many students to "see connections" or to relate ideas either within an area of mathematics or even across mathematical disciplines. For example, geometry and algebra are often viewed by high school students as totally distinct and unrelated subjects. (Rarely are students shown that traditional geometric proofs have corresponding coordinate proofs.) Such misconceptions can easily be avoided through appealing instructional activities that demonstrate the connections between ideas and between disciplines. It is important that teachers take advantage of this important value of non-text-related activities and not view them as simply activities to entertain students — to divert them from the duller, "but more important," course content.

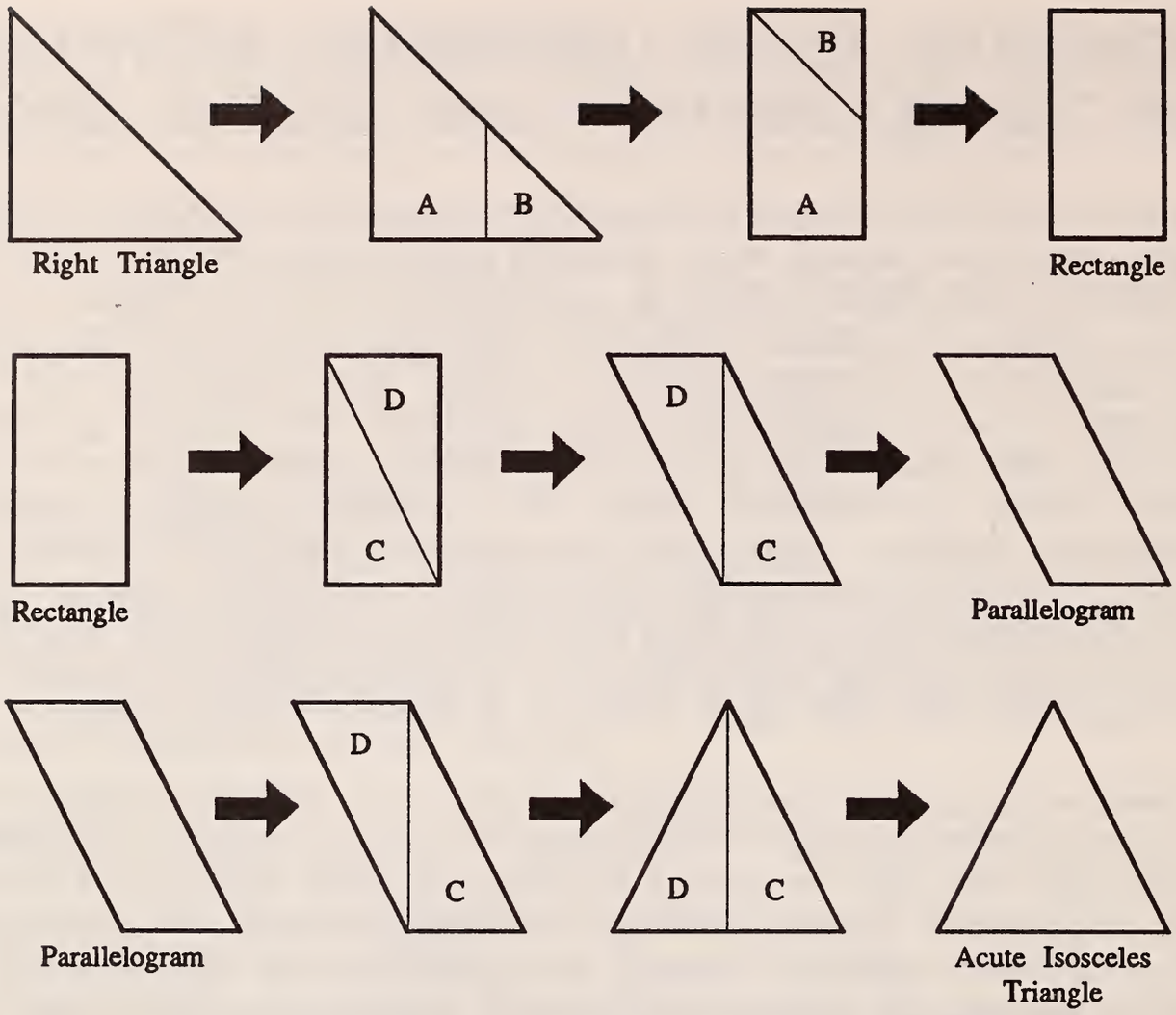
Because of the obvious progression of a learner from concrete to abstract in terms of conceptual understanding, the instructional "soundness" of using manipulatives for instruction has long been recognized. However, as stated above, the full benefit of "hands-on" instruction is seldom attained if teachers themselves do not recognize and transmit the strong connections between the activities and the concepts being taught. For many teachers, the textbook-worksheet-drill approach may seem more efficient, but its effectiveness is often superficial and shortlived. The literature on how to use tangrams, geoboards, cuisinaire rods, base-10 blocks, paper-folding, felt boards, attribute blocks, etc. is readily available. A good sourcebook for such activities is *Geometry for Grades K-6,* published by the National Council of Teachers of Mathematics, Reston, Virginia. The remainder of this chapter simply describes a few illustrative instructional activities and explains how the activities relate to important geometric concepts as well as other mathematical concepts. We also include a page of the shapes referred to, which can be copied on to board or heavy paper for cutting out and using in the classroom.



Tangrams

The seven pieces of the ancient Chinese tangram puzzle have some interesting relationships. Children can develop an intuitive sense of these relationships by using the pieces to fill in shapes. (Some suggested shapes are given here.) But there are tangram-like activities that develop students' understanding of numerical systems and logic, as well as the logical relationships among plane figures.

One suggestion for such activities comes from Rahim and Sawada in an article published in *School Science and Mathematics.* Here they suggest dissecting and rotating plane figures to make new figures which have the same area with, of course, a different shape. For example,



A right triangle transformed into a piece-wise congruent, acute isosceles triangle.

Such activities allow children to see for themselves the invariance of area and the potential for decomposition that is inherent in all figures. When using tangrams to construct various polygons, the teacher might draw students' attention to the basic qualities of the square and triangle with such questions as:

How did 'Tan' ever think of cutting up the square into these 7 pieces?

Why do these 7 pieces work so well?

Are there other ways to dissect the square that would work just as well?

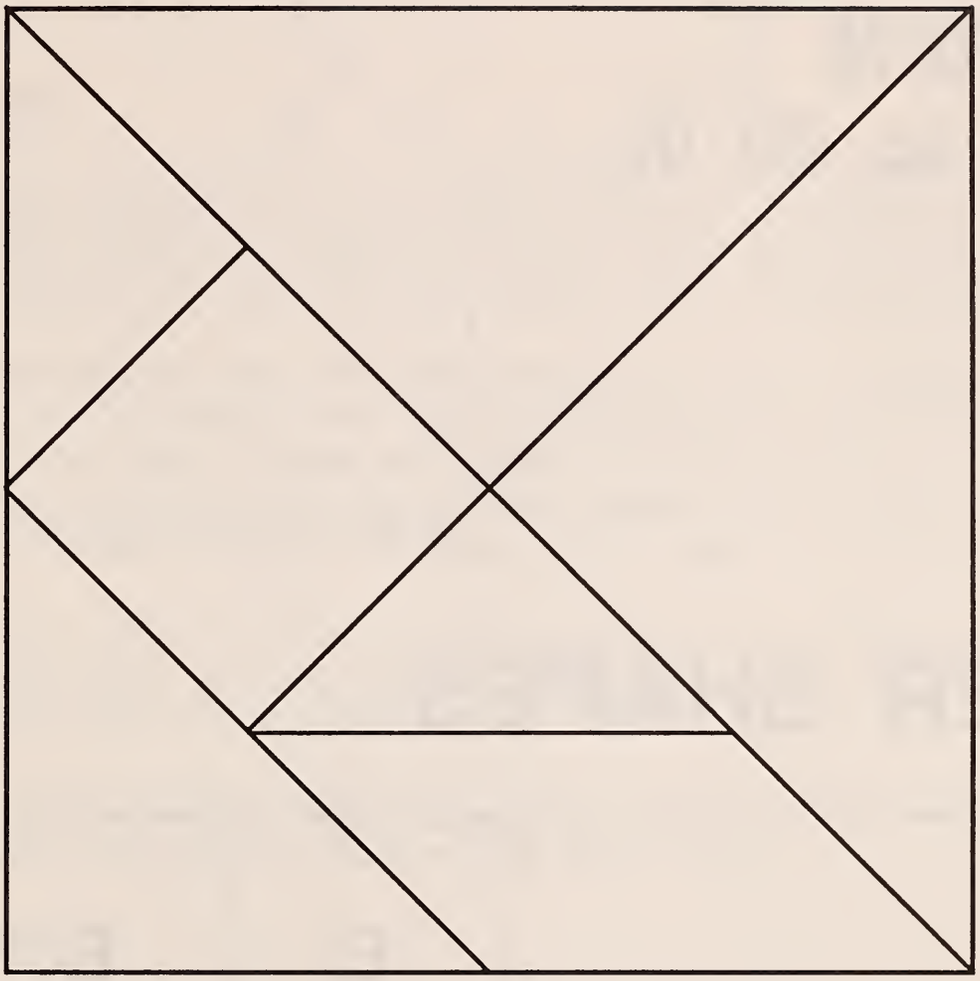
In addition, the figures can be used as visual representations of prime factorization. In this case, the teacher might ask:

Are there some particular numbers which can be used like building blocks to construct other numbers?

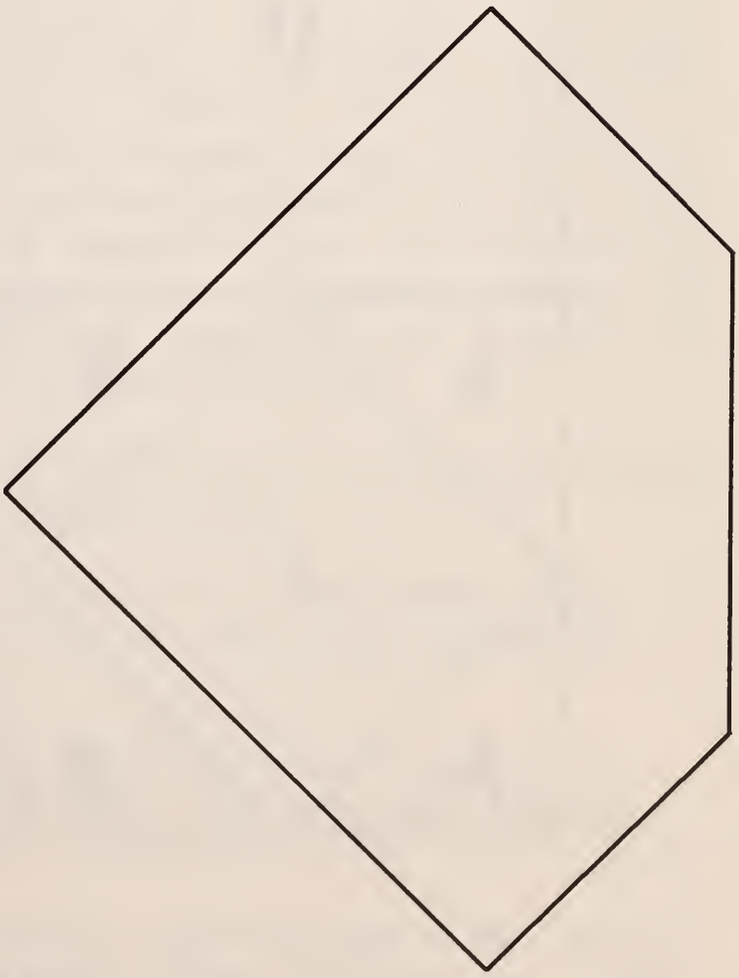
Are some numbers better building blocks than others?

Rahim, M. and Sawada, D. "Revitalizing School Geometry Through Dissection-Motion Operations." From *School Science and Mathematics*, 86 (3), 1986: 235-246.

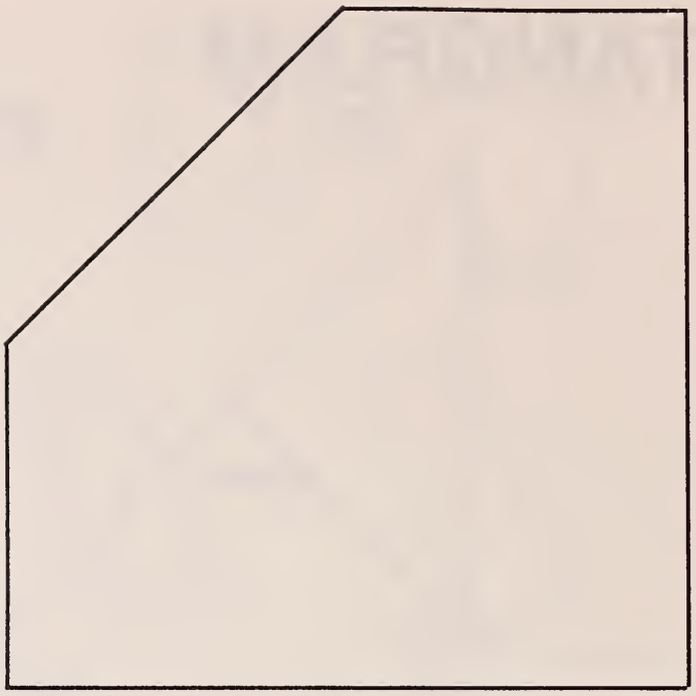
TANGRAM



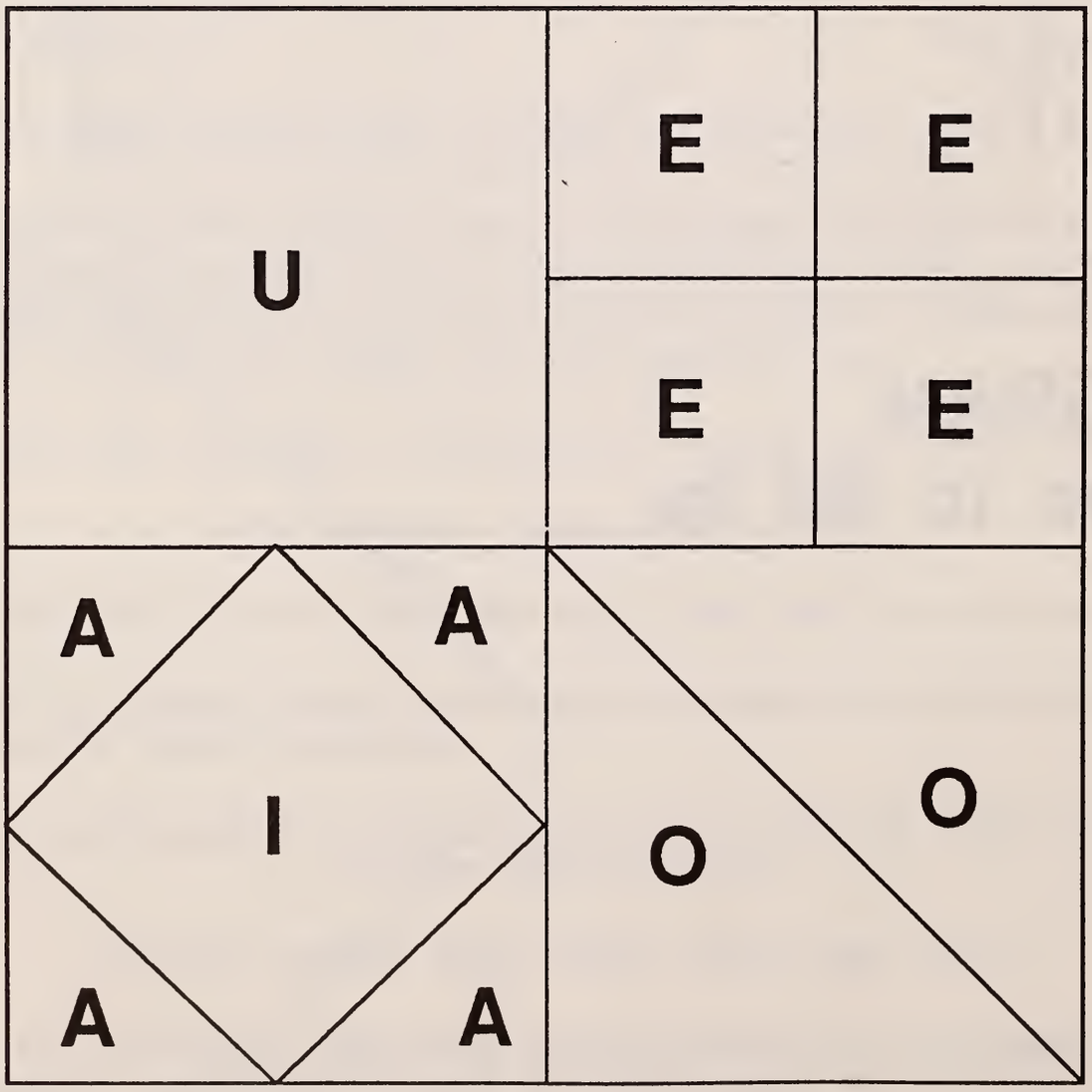
TANGRAM
Shape to fill in:



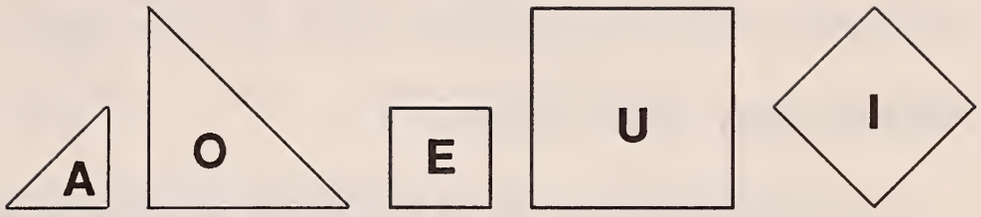
TANGRAM
Shape to fill in:



OTHER SHAPES



Other Shapes



Mary Montgomery Lindquist, in an article in *Geometry for Grades K-6,* suggests other mathematical activities for tangram-like pieces. Using a set of 12 large and small triangles and squares (see page 20), she gives a set of problems which involve:

basic relationships: How many A's cover an E? If O is 1, what number is U?
patterns: Using 2 pieces, how many different figures can you make?
logic: What is twice as large as A and smaller than I?
area: Given the outline of a figure, what pieces will cover it?
costs: If U costs 12 cents, how much will E cost?

Using these spatial shapes in different problem contexts allows children to understand not only the relationship between different geometric shapes, but the relationship between different mathematical systems.

Lindquist, M.M. "Problem Solving With Five Easy Pieces." In *Geometry for Grades K-6,* edited by Jane M. Hill. Reston, VA: NCTM, 1987.

Paper Folding

The assessment results clearly indicated students' difficulty with visualizing the result of their actions. For example, they had difficulty with recognizing the results of paper folding or folding a network to form an open box. Obviously, more activities in folding, cutting, and unfolding are needed, as well as more practical work in constructing models. For example, the building of model houses allows students opportunities to realize the need for right angles and rigid structures. When these houses are later decorated with pasted paper, measurement and area are called for, as well as other activities such as the drawing of plans and elevation drawing. Not only does mathematics become practical for these children, but such activities form a wonderful starting point for discussions and further discoveries.

Paper folding in itself is worthwhile. It is an important way for children to understand the concept of symmetry such as in the following example:

Fold a rectangular sheet of paper in half. Cut off a corner of the fold by a straight cut. Unfold the corner you have cut off and describe all the properties of the shape made.

Fold another rectangular sheet of paper in half and in half again. Cut off the corner by a straight cut, unfold, and describe the properties. How do they differ from the first?

However, paper folding has other uses as well. It can be used to help children understand the relationship among angles, as when children are set the task of listing all right angles in the classroom using a set square made from paper folded twice.

When these squares are folded again, drawn and cut, children begin to see the relationship among the resulting angles and the original whole.

Partitioning and Area

One of the frustrations of all students at all levels of mathematics is the expectation that many formulas must be accepted on faith. Some teachers are pretty good at showing the students the derivations of formulas so that they will have seen them at least once and not feel that they are so mysterious. However, allowing for a more active involvement of students in deriving formulas and giving students the capability and inclination to solve problems without having to recall formulas are desirable. No student should ever be at a loss in finding the area of a triangle because of the failure to recall the formula $A = \frac{1}{2}bh$. Finding areas by partitioning irregular shapes into more familiar pieces and summing the areas of the pieces is something that should be given a great deal more emphasis in instruction.

Handouts with irregular figures drawn on grids can be useful for in-class and homework assignments. Students should be asked to find the areas of the figures by partitioning them. These activities should not be done until the students understand the concept of area and how to find the area of rectangles and right triangles. Once they understand these, they should find the areas of parallelograms, rhombuses, trapezoids, and many irregular shapes without the benefit (or hindrance) of formulas. Eventually they might derive some formulas, or even show their derivations, but formulas should not stand in the way of understanding or even performance.

Probability as Area

The relationship between probability and area is an important concept in the study of mathematics. It comes into play in the interpretation of simple pie graphs at the lowest grade levels as well as in the understanding of important statistical concepts involving probability distributions at the higher levels.

With a little background in simple probability via coin tossing, die rolling, etc., students are ready for the "coordinate dartboard" — a square dartboard consisting of a grid. Initially, the students should be asked about the probability of a dart hitting the upper half (1/2), the second quadrant (1/4), etc. They should acknowledge that even though there are an infinite number of points a dart could strike in a quadrant, one could determine a simple probability using area. Next, upon being asked how many darts out of 20 would most likely strike in a particular quadrant (5) the students should soon recognize that

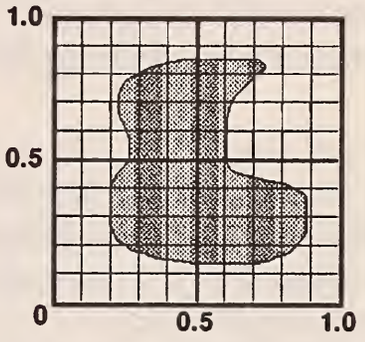
$$\frac{\text{area of quadrant}}{\text{total area of board}} \cong \frac{\text{\# darts in quadrant}}{\text{total darts thrown}}$$

At this point, regular and irregular shapes can be drawn on the dartboard, and students can be asked how to estimate the areas of the figures. "Darts" can be coordinates (ordered pairs) either written on cards and drawn from a hat or identified

using a random number table. The coordinates can identify points or squares inside or outside the figure and

$$\frac{\text{area of figure}}{\text{total area}} \cong \frac{\text{\# hits}}{\text{\# hits + \# misses}}$$

After the "dart-throwing," an area estimate is easily found from the above formula. If students are at a level where decimals are understood, the total dartboard can have length and width each equal to one so that the total area is one and the area of the figure is simply the fraction on the right side of the equation.



Students should see that finer grids would result in more accurate estimates of area. The notion that what the students are actually doing is taking a sample of points (or squares) and relying on probability should definitely be recognized by or communicated to students.

Rosenberg and Johnson. *Geometry, A Dimensional Approach*. New York: Macmillan Co., 1968.

Geometric Directions

Most teachers can admit that there are skills or concepts they did not understand as students, but which they understood well after they had to teach them to others. Translating or explaining is the main way we demonstrate conceptual understanding. The following activity capitalizes on this principle. It is an activity that should take place at the beginning and at the end of an instructional unit on or course in geometry.

Step 1: (teacher preparation) Draw a picture of a three dimensional figure — a cube works well. Then write a set of instructions giving someone the steps to follow in order to draw the figure. The directions cannot tell what the figure is or describe it. They can only tell the students where to locate points on the paper and where to draw lines. For example, "1. Locate point A on your paper one inch from the left margin of the paper and one inch from the bottom edge of the paper. 2. Draw a four inch line segment that is vertical and has point A as its bottom end point. Label its top endpoint 'B.' 3. With point B as the left endpoint draw a four inch, horizontal line segment. 4. With point B as an endpoint, draw a two inch line segment up and to the right – forming an angle of 45 degrees to the horizontal line segment drawn in step 3," and so on.

NOTE: The directions should be "perfect" — so that a student following them exactly cannot go astray and will produce a duplicate of your figure.

Step 2: In class, distribute your directions (not your drawing) to your students. Ask them to follow your directions precisely. The materials they need are a piece of paper, a pencil, a protractor and a ruler.

After an appropriate amount of time, show several students' results to the class. There should be many good drawings of your cube.

Step 3: For homework, ask the students each to draw some geometric figure consisting of line segments, preferably one with more than just a few line segments. Solid figures are nice, but plane figures with diagonals, etc. are good too. Also a part of this homework assignment is the writing of a set of instructions by each student giving directions for drawing his or her figure. The students have your "model" to help them.

Step 4: In class, have the students keep their drawings hidden but trade directions. Each student should then follow the partner's directions just as the students followed your directions the previous day. You may tell the students at the start that if some information seems vague or missing — e.g., the direction a line segment is to be drawn is not stated — then they can do what they want, still following the part of each direction that they can — e.g., draw the line segment in any direction they want.

Step 5: After an appropriate amount of time, have the original drawings taken out and paired up with their supposed duplicates. It is show and tell time! A lot of the drawings will be pure scribble, and should be quite entertaining — the matched pairs of drawings worthy of posting on the bulletin board.

Step 6: After the unit of instruction (assuming it is a substantial unit) or at the end of the course, the same activity should be repeated. The improvement in the directions and resulting drawings will be remarkable. The students will demonstrate that they have a better understanding of the preciseness demanded in geometry as well as many geometry concepts (horizontal, parallel, midpoints, etc.) and have developed many measurement skills. (The older students, of course, could use degree measures in their instructions for directions in which to draw lines.)

Comments

Geometry (and in fact, mathematics in general) lends itself particularly well to activity-oriented instruction. Furthermore, the ideas for activities are quite accessible — the Golden Section/Divine Proportion, Fibanacci sequence, Pascal's Triangle, ancient Greeks' "solutions" to the trisection problem, and so on. When used to teach concepts rather than to merely entertain, such activities can be invaluable. Most teachers have heard, "I hear and I forget; I see and I remember; I do and I understand." Yet so often, the third part of the maxim gets shortchanged when we teach mathematics. The best thing we, as teachers, can do to promote understanding of mathematics is to *allow* students to learn — let *them* discover relationships, derive formulas, generate data to use. They can do that best when they can visualize and manipulate real objects. Geometry is more than a mathematical topic — it is a vehicle for understanding.

CHAPTER FOUR: TEACHING GEOMETRY IN THE SECONDARY SCHOOL

by
Thomas J. Cooney
Douglas Jones
University of Georgia

Because secondary school geometry is both visual and axiomatic it provides a unique context for a variety of mathematical activity including the following: visualizing, reasoning, constructing figures, applying geometric concepts to solve problems, and developing an appreciation for the historical development of geometry. Studying secondary school geometry should not be synonomous with doing proofs to the exclusion of other important mathematical activity. Recent research based on the van Hiele levels of geometric understanding suggests that students exit secondary school geometry with only minimal competence for writing proofs. Further, the average ninth or tenth grader's level of geometric understanding is usually at the level of recognizing and applying properties of figures and is generally not at the level of understanding the axiomatic nature of geometry. This situation is often reflected in students' legendary denouncements of proofs in particular and sometimes geometry more generally when the majority of classtime is spent proving theorems. The solution is not to completely dismantle geometry as an axiomatic system but rather to emphasize other outcomes which are essential for the study of more advanced mathematics and for applying mathematics to "real life" situations. The following sections suggest a variety of activities that promote a diversified perspective about geometry.

Visualizing

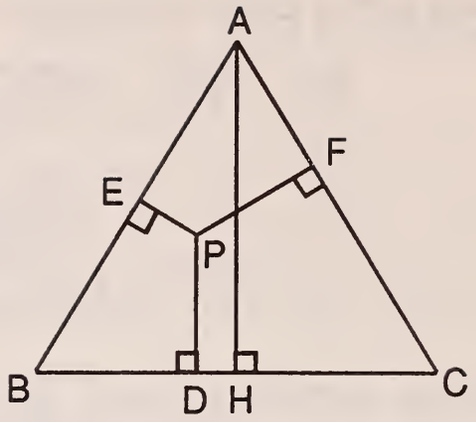
- Visualizing activities can be developed in several ways. One such activity involves cross-sectional cuts of three dimensional figures. For example, consider the cube and the following questions.

    1. Describe or sketch a cross-section of a cube that gives a square.
    2. Describe a cross-section of a cube that gives a rectangle.
    3. Is it possible to have a cross-section of a cube that gives an isosceles triangle? An equilateral triangle?

  Cross-sections of rectangular prisms (that are not cubes) and of other three dimensional figures can also be considered. Students can construct models of the space figures out of Styrofoam and experiment with cutting the models to investigate various cross-sections.

- A different type of visualizing activity requires students to transform a statement about a two-space situation into a statement about an analogous three-space situation. Consider the following statement.

If P is a point in the interior of equilateral triangle ABC, and $\overline{PD}$, $\overline{PE}$, and $\overline{PF}$ are segments perpendicular to the sides of the triangle, then PD + PE + PF = AH, where $\overline{AH}$ is an altitude of the triangle.



Once students have verified this statement (perhaps by empirical means or by plausible reasoning — such as considering the three perpendicular segments when P is close to vertex A), they can examine the analogous three-space problem in which point P lies in the interior of a regular tetrahedron.

- Locus problems can also be a source of visualizing activities. For example, students can be asked to consider the statements below which involve points in a plane.

    1. Describe the set of points in a plane that are 3 cm. from a point P in the plane.
    2. Describe the set of points in a plane that are 3 cm. from line *m* in the plane.
    3. Describe the set of points in a plane that are equidistant from two intersecting lines *p* and *q*, and are 3 cm. from the point of intersection of *p* and *q*.

Students can then consider the same conditions and describe or sketch the locus of points in a three-dimensional setting.

Reasoning as a Precursor to Understanding Logical Arguments and for Constructing Proofs.

- Deductive reasoning is a central theme throughout mathematics. Students need experience in analyzing "If . . . , then . . ." statements. As part of the experience of working with "If . . . , then . . ." statements, students are often asked to form converses, inverses, and contrapositives of a given statement. Statements that appear in advertisements are readily available. Some of the statements are explicitly in the form "If . . . , then . . ."; others are more implicit or subtle. Consider the following statements.

> If you'd rather not wait 'til the early 1990s,
> we're here now with the vehicles they're
> hoping to beat. (*Advertisement for Chrysler Motors*)

or,

If you can't tell the difference, why pay more?
(*Advertisement for Tyco Super Blocks*)

or,

With Pacquin, soft beautiful skin was never expensive.

The latter advertisement suggests the following "If . . . , then . . ." statement.

> If you use Pacquin, then you will have soft, beautiful skin, and it will not be expensive.

Using these statements, students can form the converse, inverse, and contrapositive statements, and discuss which of them are necessarily true if one assumes the given statement is true.*

- Inductive reasoning also plays an important role in studying geometry. Students can use inductive reasoning to discover geometric relationships or theorems and thereby develop a sense of "ownership" of a conjecture before it is proven. The statement given earlier about an equilateral triangle is an excellent candidate for discovery using inductive reasoning. By considering different locations for the interior point and the sums of the lengths of the three perpendicular segments from those points to the sides of the triangle, students can discover that (a) the sum is always a constant and (b) that the sum is equal to the length of the altitude.

- In another context students can discover the relationship between the number of sides of a polygon and the maximum number of diagonals of the polygon by drawing figures, counting the diagonals, and constructing a chart similar to the one below.

Number of sides	Number of diagonals
3	0
4	2
5	5
6	9

Algebra can be used to derive a formula that relates that number of sides and the number of diagonals, or students may discover a formula by considering more cases. The relationship between the number of sides of a polygon and the sum of the measures of its interior angles can also be discovered through the use of inductive reasoning. Developing a formula or writing a formal proof should not be the ultimate goal for all. For some students, it may be sufficient to see that a relationship exists.

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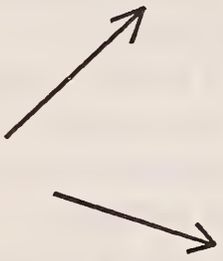
* Students' difficulty with this kind of analysis is discussed on page 11.

Constructing Figures

Activities involving compass and straightedge constructions can be used as a vehicle for using inductive reasoning. Such constructions can provide a means for investigating the concurrency of altitudes, medians, angle bisectors, and perpendicular bisectors of the sides of triangles. Constructions can also be used to help students synthesize and apply many geometric properties. Consider, for example, the information that must be used in order to construct a square if one is given a segment that represents the sum of the lengths of a diagonal and a side.

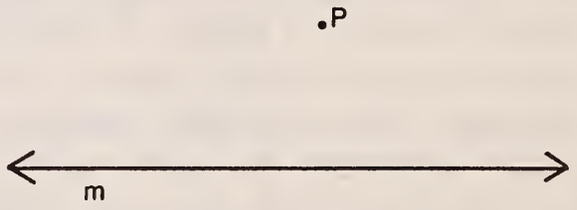
Many properties can be discovered using constructions that might not otherwise be attainable by students. For example, a student might investigate the collinearity of the circumcenter, the orthocenter, and the centroid of a given triangle using constructions. Other constructions can involve basic constructions but with special restrictions. Consider the problems below:

(1)



Construct the bisector of an angle given the rays but not the vertex.

(2)



Construct the perpendicular from P to *m* without striking arcs below *m*.

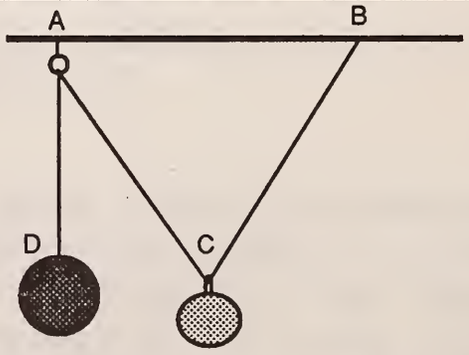
Applying Concepts to Solve Problems

- Geometry is also a rich area for solving application problems. It is important that students see connections between geometry and the world in which they live. The following two problems involve the use of geometric concepts in a practical context.

(1)

The opening in an hourglass is 1 mm. in diameter. What size must the opening be if the sand is to pass through twice as quickly?

(2)



The weight at D is attached to a pulley with a loop at point A. Point B is a fixed point. Triangle ABC is an equilateral triangle with each side having a length of 3 feet. How far would point C be raised if D is pulled down 1 foot?

- There is another kind of application of geometric concepts that students should encounter, namely, the use of geometry to interpret and give meaning to algebraic concepts. Inducing properties of linear or quadratic equations from their graphs is one obvious application of geometry. Other applications may lie closer to

definitions encountered in algebra classes. For example, interpreting $|2x - 3|$ as the *distance* of the quantity $2x - 3$ from zero may add a visual component to a student's understanding of absolute value that a more algebraic definition [$|2x - 3| = 2x - 3$, if $2x - 3 > 0$ but $= -(2x - 3)$, if $2x - 3 < 0$] does not.

Appreciating Historical Developments

Finally, activities can be created that emphasize an appreciation of the historical development of geometry. A particular case in point is the historical circumstances surrounding the possibility of proving the parallel postulate. If a class follows the sequence of first studying absolute geometry and then the parallel postulate (i.e., studying the parallel postulate after the study of congruent triangles), then the significance of the parallel postulate can be considered.

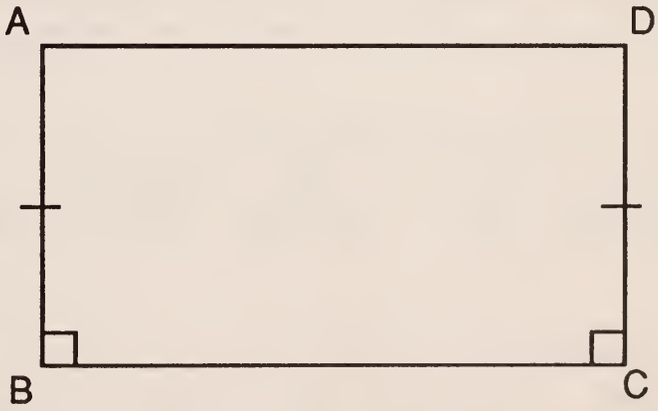
Although the statement of the parallel postulate varies from text to text, the postulate usually is represented by a statement similar to the following:

> For every line *l* and for every point P that does not lie on *l*, there exists a unique line *m* through P that is parallel to *l*.

Students can be asked to identify those theorems whose truth depends upon the parallel postulate, such as theorems about the measures of the angles of triangles or polygons dependent upon this postulate.

Students could also study the mathematicians who investigated the consequences of accepting or rejecting the parallel postulate. One of the mathematicians who tried erroneously to prove the postulate was the Italian Jesuit priest Girolamo Saccheri (1667-1733). His attempt, one of many by ancient mathematicians, involved what is now called the Saccheri quadrilateral.

A Saccheri quadrilateral has right angles at its base and two congruent sides as shown.



Able high school students can prove, as did Saccheri, that the summit angles ($\angle A$ and $\angle D$) are congruent by drawing auxiliary lines and proving triangles congruent. Saccheri also tried to prove that the summit angles are right angles without using the parallel postulate or its equivalent. He never realized the construction of this proof is not possible. Students can also find biographical information about Bolyai, Lobachevsky, and Gauss, all of whom were principal players, and report to their classmates on some of the human drama in which these famous mathematicians were involved.

Conclusion

These suggested activities are aimed at promoting a balance of mathematical outcomes that include much more than just proof-oriented objectives. The study of mathematics, even by mathematicians, consists of many activities including the ones suggested here. Although perhaps not all students can profit from the specific activities suggested in the preceding sections, every student can profit from activities involving the processes of visualizing, reasoning — both deductive and inductive — constructing figures, applying concepts to solve problems, and developing an appreciation of the development of geometry. No course in geometry should exclude these activities on the basis of student abilities. Instead, activities should be created that reflect these general processes that are appropriate for students of all ability levels.

CHAPTER FIVE:
FINDINGS & RECOMMENDATIONS

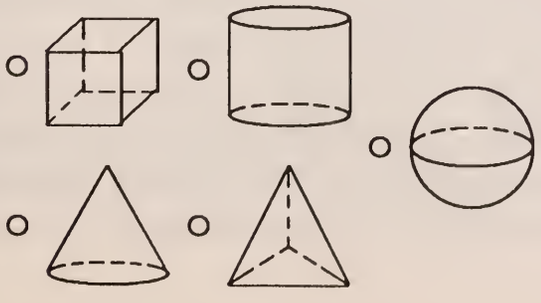
I. Findings

1. The relevance of geometry to the rest of the mathematical curriculum is not appreciated. Too often, geometry is looked upon as an add-on, something to be taught at the end of the term if there is time enough for it.

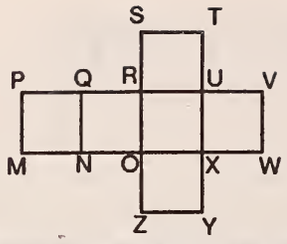
As this report has attempted to illustrate, geometric ideas and components infuse the mathematics curriculum and can be looked upon as the unifying theme for many mathematical concepts and procedures. Even more important, geometry is an area in which the development and systematization of ideas strongly parallels our own cognitive development. Its study begins with concrete objects that are of interest in themselves. It is through manipulating these objects — transforming them and observing the results of our actions on them — that we begin to induce their properties. What makes a square? How are triangles and rectangles related? What are the variables and the invariances that define the concepts? In turn, we use these abstractions for the content of deductive argument.

If geometric ideas pervade the mathematics curriculum, if geometric thinking is logical and developmental, why do students perform poorly — not only at the highest level of deductive thinking, but at the more basic levels of abstraction? We can only suggest that the subject matter is not viewed as a major priority in the mathematics curriculum. Our basis for this conclusion comes from test results. We assume that, as a general rule, the opportunity to learn is reflected in performance. It also comes from interviews with teachers and from a review of the more commonly available texts. But we have more direct evidence. When presented with test items that involved geometric ideas and asked if their students had been taught the necessary information to answer them, third and seventh grade teachers were not generally positive. (See below.)

Grade 3

	Yes	No, taught previous year	No, to be taught later this year	No, to be taught in a later year	No, not in curriculum
What is the area of the shaded figure? ○ 16 ○ 17 ○ 18 ○ 19 ○ 20	18%	1%	34%	38%	10%
Which figure shows a CYLINDER? 	28%	4%	42%	23%	4%

Grade 7



The diagram shows a cardboard cube which has been cut along some edges and folded out flat. If it is folded to again make the cube, which two corners will touch corner P?

A. corners Q and S
B. corners T and Y
C. corners W and Y
D. corners T and V
E. corners U and Y

Yes	No, taught previous year	No, to be taught later this year	No, to be taught in a later year	No, not in curriculum
17%	3%	23%	25%	31%

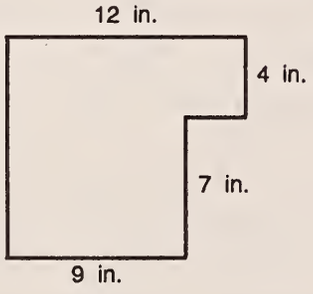
Which of the following is true for all parallelograms?

A. Opposite angles are equal.
B. All sides are equal.
C. The diagonals are equal.
D. Adjacent sides are perpendicular.

Yes	No, taught previous year	No, to be taught later this year	No, to be taught in a later year	No, not in curriculum
45%	3%	31%	17%	5%

What is the area of this figure?

A. 46 inches
B. 111 square inches
C. 2 feet, 8 inches
D. 23 square inches



Yes	No, taught previous year	No, to be taught later this year	No, to be taught in a later year	No, not in curriculum
40%	4%	40%	14%	2%

Perhaps of more general concern is the evidence that, when such topics are taught at the middle school level, teachers view them as suitable primarily for their more advanced students. The summary figures mask the fact that these activities were 50% more likely to be taught in advanced classes than in remedial ones. Without more detailed questioning, it is impossible to know precisely what geometric topics *are* being taught to various groups of students in the schools. However, the assessment results do suggest that geometry and measurement are not considered important elements of mathematics education.

The opportunity that measurement affords in reinforcing geometric concepts and offering a practical application for geometric learning is often ignored. Furthermore, this lack of recognition of the relationship of geometry to other areas of mathematics is not only true of measurement. Other applications of geometric concepts are likewise ignored. The fact that geometric ideas permeate so much of later work in calculus, analytic algebra and trigonometry, as well as forming the content for deductive logic, is often ignored. Perhaps, most important, few make explicit geometry's parallelism to the numerical system. And yet, the laws of associativity and commutivity that both share can serve to reinforce both areas and help students' understanding of the unity and rationality of mathematics itself.

2. Throughout schooling, there is little recognition of the internal development of geometric thinking and even less attempt to incorporate this developmental model into the curriculum.

Geometry is a way of ordering the objects that fill the world of the young — labels, packages, balloons and lollipops. By observing the forms of these objects, by comparing them, taking them apart and putting them together, children begin to form intuitive understandings of the meaning of rectangularity, circularity, squareness. They begin to recognize not only prototypical examples of squares and triangles, but how they relate to one another and how they are defined.

Geometry is seldom approached from this standpoint. In the early grades, spatial activities are considered recreational, enrichment activities focusing on an interesting puzzle of a spatial sort. Later, most instruction is definitional. Children are taught to recognize and define. They are seldom asked to *do*. This stress on the symbolic before the visual is, as one educator remarked, "like asking someone to read before they can talk." The result is that, when asked to go beyond what has been explicitly taught, students cannot reason because they have little intuitive understanding of the basic concepts involved. More often than not, students enter high school without a firm, intuitive understanding of the concepts which they will encounter, not only in geometric work, but in other mathematical work as well.

3. Geometry at the secondary level is often seen as irrelevant and too difficult by most students.

There is evidence to support students' perceptions. It has been clearly demonstrated recently that even among senior high school students who receive extensive mathematics training, two-thirds do not understand what a mathematical proof is.

Undoubtedly, students' earlier preparation contributes to the difficulty they find in high school geometry. However, there are questions about the course itself: whether it is accomplishing its primary goal, the development and appreciation of deductive thinking. As one educator remarked, "That the traditional emphasis in Euclidean geometry is on proof goes without saying. But whether it is about *proving* is another matter."

Much of the criticism of geometry teaching at the secondary level centers on method rather than content. A desire for rigor and exactness of language, an excessive sensitivity to "sloppy mathematics" seems to dominate the subject, obscuring the fact that doing mathematics means exploring, conjecturing, trying and failing. The means of teaching a proof have little to do with how mathematicians use a proof. In mathematics, proof comes after insight, when the mathematician asks, "Now, how can I prove it?" Denying students the opportunity to explore the interesting question, to conjecture, to experiment, to "play around," denies them a mathematical experience and tells them that there is no reason to believe that a theorem is true other than accepting the teacher's authority.

Instead, proving and proof should be the occasion for discussion, an opportunity for students to reason and argue, to become aware of their own assumptions and those of others. The context may be geometry, but the intellectual activity has much broader implications. Furthermore, geometry instruction itself need not be confined to axioms. Visualizing, constructing, and applying spatial concepts to solve problems are important mathematical activities and the stuff of hard and productive thinking, as suggested in Chapters Three and Four.

4. Teachers lack geometric training.

"The lack of success that characterizes so many students' experiences in geometry discourages other students from taking it, keeps elementary school teachers from wanting to take geometry in college or teach it to their students, and perpetuates the cycle of poor performance."*

Since geometry is seldom treated as a unified subject firmly based in the physical world around us, teachers do not often view it in these terms. More likely they think of it as the geometry they learned as immature adolescents; or, if they did take a geometry course in college (and few do), their experience was highly axiomatic. As a consequence, for teachers in the elementary grades, tangrams, paper folding and shape matching are seen as busy work compared to the serious business of computation. Later, geometry is seen as definitional and formula-driven, a body of knowledge handed down from a higher authority.

On the other hand, one cannot blame teachers for their inability to make judgments about the relevance of individual topics of activities in an area where they are unprepared. Many do not feel comfortable enough with the subject area to allow their students the exploration and discussion that is so often advocated. As a result, these teachers stay close to the book and demand the one correct answer. As a further result, the understanding of their students suffers and the cycle begins again.

Unfortunately, there are few resources for teachers who wish to extend their understanding of geometry. For elementary teachers, few courses deal with geometric understanding rather than geometric activities. For the middle and secondary level teacher, courses that exist sacrifice pedagogy for advanced mathematics. In neither case is the teacher prepared for teaching geometry as a productive and exciting mathematical activity. As a result, until there is a concerted effort to engage teachers — existing or prospective — geometry will remain a puzzle or a hardship to most who encounter it.

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* Zalman Usiskin. In *Learning and Teaching Geometry, K-12.* Reston, VA: NCTM, 1987.

II. Recommendations for others

1. Schools should evaluate their geometry/measurement curriculum in light of the developmental demands of the concepts involved for alignment throughout the grades. Instruction should be built on a natural progression of the recognition of spatial figures, the understanding of their properties and the use of these properties in logical reasoning and analysis.

2. Teachers of geometry at all levels need to enrich and to structure the spatial experiences of their students. Physical activities and exploration should not be confined to the early grades but should form an intregal part of all geometry instruction. Concrete aids such as geoboards, as well as computer-based materials, should be used in all classes.

3. Discussion should be recognized as a primary vehicle for the development of geometric thinking. At the early levels, students should be encouraged to talk about the spatial and geometric activities they have been engaged in. At the upper grades, they should be engaged in cooperative experiences which lead to different conjectures by different students and the resolution of conflicts by argument and evidence.

4. All K-6 teachers should be required to study geometry concepts and teaching methods. All middle and secondary level mathematics teachers should be required to study geometry as part of their mathematics curriculum at the college level.

5. School districts should take responsibility for the expansion and improvement of geometry teaching by implementing professional development programs in geometry concepts and teaching methods.

6. Finally, students should not be kept from studying geometry merely because they are poor in arithmetic or algebra. Instead, geometry should be considered a basic theme in the mathematics curriculum to be taught and learned by all students.

To quote one mathematics educator:

> "It is as if we say, *'Because you are not so good at basketball, we won't let you bowl either.*' Too often we find out what the student cannot do and then give the student no opportunity to show what he or she can do."
>
> — *(Zalman Usiskin)*

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